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Asymptotics for the ${}_4F_3$ Polynomials

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Asymptotic expansions are given for the ${}_4F_3$ and ${}_4\phi_3$ orthogonal polynomials which generalize the classical orthogonal polynomials. The expansions are applied to determine the complex sets of convergence of series of these polynomials. The proof of the main asymptotic expansion uses a convexity argument which is especially well suited to estimating certain hypergeometric series and their integral analogs. As an alternative approach to the asymptotics, a uniform version of Darboux's method is described. © 1991 Academic Press, Inc.

1. INTRODUCTION

The polynomials

$$P_n(z^2) = (a+b)_n\,(a+c)_n\,(a+d)_n \times {}_4F_3\left[\begin{matrix} -n, n+a+b+c+d-1, a+iz, a-iz; 1 \\ a+b, a+c, a+d \end{matrix}\right] \tag{1.1}$$

(polynomials of degree n in the variable z^2) generalize the classical orthogonal polynomials as well as the $6-j$ symbols of angular momentum. They satisfy various orthogonality relations, depending on the values of the parameters a, b, c, d [10].

If a, b, c, d are all positive except for complex conjugate pairs with positive real parts, then $P_n(x)$ is real for real x, and

$$\int_0^\infty P_m(x)\,P_n(x)\,w(x)\,dx = \delta_{m,n} h_n \tag{1.2}$$

with

$$w(x) = \frac{\left(\dfrac{\Gamma(a+b+c+d)\,\mathrm{sh}(2\pi\sqrt{x})}{|\Gamma(a+i\sqrt{x})\,\Gamma(b+i\sqrt{x})\,\Gamma(c+i\sqrt{x})\,\Gamma(d+i\sqrt{x})|^2}\right)}{2\pi^2\Gamma(a+b)\,\Gamma(a+c)\,\Gamma(a+d)\,\Gamma(b+c)\,\Gamma(b+d)\,\Gamma(c+d)}$$

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and

$$h_n = \frac{n!(n+a+b+c+d-1)_n\,(a+b)_n\,(a+c)_n \cdots (c+d)_n}{(a+b+c+d)_{2n}}.$$

If a is negative while $a+b$, $a+c$, $a+d$ have positive real part, then the orthogonality relation (1.2) must be modified by the addition of the finite sum

$$\sigma \sum_{\substack{k \geqslant 0 \\ a+k<0}} \mu_k\, p_m(-(a+k)^2)\, p_n(-(a+k)^2) \tag{1.3}$$

to the left-hand side. Here

$$\sigma = \frac{\Gamma(b-a)\,\Gamma(c-a)\,\Gamma(d-a)\,\Gamma(a+b+c+d)}{\Gamma(b+c)\,\Gamma(b+d)\,\Gamma(c+d)\,\Gamma(-2a)}$$

and

$$\mu_k = \frac{(2a)_k\,(a+1)_k\,(a+b)_k\,(a+c)_k\,(a+d)_k}{k!(a)_k\,(a-b+1)_k\,(a-c+1)_k\,(a-d+1)_k}.$$

There is also a purely discrete orthogonality relation. If $a+b=-N$, N a nonnegative integer, then

$$\sigma \sum_{k=0}^{N} \mu_k\, p_m(-(a+k)^2)\, p_n(-(a+k)^2) = \delta_{m,n} h_n, \tag{1.4}$$

where

$$\sigma = (a-c+1)_N\,(a-d+1)_N/(2a+1)_N\,(1-c-d)_N$$

(μ_k and h_n as above).

In Section 2, we derive asymptotic expansions for $P_n(z^2)$ as $n \to \infty$. These contain, for example, the following estimates. For $z^2 \notin \{-m^2/4 : m \in \mathbb{Z}\}$

$$P_n(z^2) = \frac{(a-iz)_n\,(b-iz)_n\,(c-iz)_n\,(d-iz)_n}{(-2iz)_n}\,[1+O(n^{-2})]$$
$$+\frac{(a+iz)_n\,(b+iz)_n\,(c+iz)_n\,(d+iz)_n}{(2iz)_n}\,[1+O(n^{-2})].$$

Accordingly, if $x>0$ and the parameters satisfy the conditions for the real orthogonality relation (1.2), then

$$P_n(x) = C_n\,[2\,|A(i\sqrt{x})|\,\cos(2\sqrt{x}\ln n - \arg A(i\sqrt{x})) + O(n^{-1})],$$

where

$$C_n = (2\pi)^{3/2} e^{-3n} n^{3n+a+b+c+d-3/2}$$

and

$$A(z) = \Gamma(2z)/\Gamma(a+z)\,\Gamma(b+z)\,\Gamma(c+z)\,\Gamma(d+z). \tag{1.5}$$

If $\operatorname{Im} z > 0$ and $A(-iz) \neq 0$, then

$$P_n(z^2) \sim A(-iz)\, C_n n^{-2iz}$$

(even if $z^2 = -m^2/4$). If $\operatorname{Im} z > 0$, $A(-iz) = 0$, and $A(iz) \neq 0$ (this case corresponds to a mass point in (1.3)), then

$$P_n(z^2) \sim A(iz)\, C_n n^{2iz}$$

except that the right hand side must be doubled if $z^2 = -m^2/4$. If $A(-iz) = A(iz) = 0$, then $P_n(z^2)$ is zero for all n sufficiently large. This case corresponds to a mass point in the discrete orthogonality relation (1.4).

In the cases where $\{P_n(x)\}$ is orthogonal with respect to a positive measure, the asymptotic formulas for the orthonormal polynomials obtained by rescaling $P_n(x)$ have C_n replaced by $n^{-1/2}$ times a factor depending only on the parameters a, b, c, d. The standardization used here has the advantage of generality—it makes $P_n(z^2)$ an entire (actually, a polynomial) function of the parameters.

The asymptotic expansions allow us to compare (in Section 3) any series $\sum a_n P_n(z^2)$ with a Dirichlet series $\sum b_n n^{-2iz}$ and conclude that the sets of convergence are parabolic regions (in the z^2-plane) along with (possibly) finitely many other points corresponding to point masses in the orthogonality relations.

The polynomials $P_n(z^2)$ are a limiting case of the polynomials [1]

$$\begin{aligned} p_n((z+z^{-1})/2) \\ &= a^{-n}(ab;q)_n\,(ac;q)_n\,(ad;q)_n\; {}_4\phi_3\left[\begin{matrix} q^{-n}, q^{n-1}abcd, az, a/z; q, q \\ ab, ac, ad \end{matrix}\right]. \end{aligned} \tag{1.6}$$

Corresponding results for these polynomials are given in Section 4. Some of these results made a more timely appearance in [6].

2. Asymptotic Expansions

The polynomials $P_n(z^2)$ have many other hypergeometric representations, obtainable by applying (and iterating) Whipple's identities [2, pp. 55, 56]

$$
\begin{aligned}
{}_4F_3\left[\begin{matrix} a, b, c, -n; 1 \\ d, e, f \end{matrix}\right] &= \frac{(e-a)_n\,(f-a)_n}{(e)_n\,(f)_n} \\
&\quad \times {}_4F_3\left[\begin{matrix} a, d-b, d-c, -n; 1 \\ d, d+e-b-c, d+f-b-c \end{matrix}\right] \\
&= \frac{(u-b)_n\,(u-c)_n}{(u)_n\,(u-b-c)_n} \\
&\quad \times {}_7F_6\left[\begin{matrix} u-1, (u+1)/2, u-e, u-f, b, c, -n; 1 \\ (u-1)/2, e, f, u-b, u-c, u+n \end{matrix}\right]
\end{aligned} \tag{2.1}
$$

valid when $d+e+f=a+b+c-n+1$ and $u=e+f-a$. Two such representations are used in this section. One is

$$
\begin{aligned}
P_n(z^2) &= \frac{\pi_n(-iz)}{(-2iz)_n} \\
&\quad \times {}_7F_6\left[\begin{matrix} 2iz-n, iz-n/2+1, a+iz, b+iz, c+iz, d+iz, -n; 1 \\ iz-n/2, iz-n+1-a, ..., iz-n+1-d, 1+2iz \end{matrix}\right] \\
&= n! \sum_{k=0}^{n} u_k(iz)\, u_{n-k}(-iz) \cdot \frac{2iz-n+2k}{2iz}
\end{aligned} \tag{2.2}
$$

with

$$
\pi_k(z) = (a+z)_k\,(b+z)_k\,(c+z)_k\,(d+z)_k
$$

and

$$
u_k(z) = \pi_k(z)/k!(1+2z)_k.
$$

Thus the ${}_4F_3$ polynomials are ${}_7F_6$ polynomials as well. Note that they also have been found to be denominators in Padé approximants to a ${}_7F_6$ [5].

The other representation is

$$
\begin{aligned}
P_n(z^2) &= (a+b)_n\,(c-iz)_n\,(d-iz)_n \\
&\quad \times {}_4F_3\left[\begin{matrix} -n, -c-d+n+1, a+iz, b+iz; 1 \\ a+b, -c+iz-n+1, -d+iz-n+1 \end{matrix}\right] \\
&= (a+b)_n\,(c+d)_n\, n! \\
&\quad \times \sum_{k=0}^{n} \frac{(a+iz)_k\,(b+iz)_k}{(a+b)_k\, k!} \frac{(c-iz)_{n-k}\,(d-iz)_{n-k}}{(c+d)_{n-k}\,(n-k)!}.
\end{aligned} \tag{2.3}
$$

This formula immediately gives a generating function

$$\sum_{n=0}^{\infty} \frac{P_n(z^2)w^n}{(a+b)_n\,(c+d)_n\,n!} = {}_2F_1\left[\begin{matrix} a+iz,\, b+iz;\, w \\ a+b \end{matrix}\right] {}_2F_1\left[\begin{matrix} c-iz,\, d-iz;\, w \\ c+d \end{matrix}\right]. \tag{2.4}$$

Our main result is that the sum (2.2) has the character of an asymptotic expansion, with the terms near the ends being the most significant. More precisely,

THEOREM. *If $z \notin \{im/2 : m \in \mathbb{Z}\}$ then as $n \to \infty$,*

$$\begin{aligned} P_n(z^2) = n! \sum_{k=0}^{r-1} u_k(iz)\, u_{n-k}(-iz) \frac{2iz-n+2k}{2iz} \\ + \pi_r(iz)\, A(-iz)\, C_n O(n^{-2iz-2r}) \\ + n! \sum_{k=0}^{s-1} u_{n-k}(iz)\, u_k(-iz) \frac{2iz+n-2k}{2iz} \\ + \pi_s(-iz)\, A(iz)\, C_n O(n^{2iz-2s}) \end{aligned} \tag{2.5}$$

with C_n, $A(z)$, $\pi_k(z)$, and $u_k(z)$ as in (1.5) and (2.2). The error estimates are uniform for a, b, c, d, and z in compact sets (avoiding points $z = im/2$).

Proof. The O-estimates in the following are intended as uniform for (a, b, c, d, z) belonging to any compact set F in $\mathbf{C}^5$ avoiding points where $z = im/2$.

When n is large, the difference between $P_n(z^2)$ and the approximation in (2.5) is (according to (2.2))

$$E_{r,s}(n) = n! \sum_{k=r}^{n-s} u_k(iz)\, u_{n-k}(-iz) \cdot \frac{2iz-n+2k}{2iz}.$$

Now, at least if r and s are large enough (depending on F), we may write

$$\begin{aligned} E_{r,s}(n) &= 2iz A(iz)\, A(-iz)\, n! \sum_{k=r}^{n-s} v_k^+ v_{n-k}^-(n-2iz-2k) \\ &= A(iz)\, A(-iz)\, n!\, O(n) \cdot \sum_{k=r}^{n-s} |v_k^+ v_{n-k}^-| \end{aligned}$$

as $n \to \infty$, with

$$v_k^\pm = \Gamma(a \pm iz + k) \cdots \Gamma(d \pm iz + k)/k!\, \Gamma(1 \pm 2iz + k).$$

To make further progress, we need to verify that, for large values of k, $|v_k^{\pm}|$ is a log convex function of k. With Δ denoting the forward difference in the variable k, we calculate

$$\begin{aligned}\Delta^2 \ln |v_k^{\pm}| &= \ln |v_{k+2}^{\pm} v_k^{\pm}/(v_{k+1}^{\pm})^2| \\ &= \ln \left| \frac{(a \pm iz + k + 1) \cdots (d \pm iz + k + 1)(1 + k)(1 \pm 2iz + k)}{(a \pm iz + k) \cdots (d \pm iz + k)(2 + k)(2 \pm 2iz + k)} \right| \\ &= 2k^{-1} + O(k^{-2}) > 0\end{aligned}$$

for k large (depending on F). Now, from log convexity of $|v_k^{\pm}|$ follows log convexity of $|v_k^+ v_{n-k}^-|$ for $r \leqslant k \leqslant n-s$, at least when r, s are both large, $n \geqslant r+s$.

By convexity, the mean value

$$\frac{1}{n-s-r+1} \sum_{k=r}^{n-s} |v_k^+ v_{n-k}^-|$$

is less than the mean of the first and last terms.

This gives

$$E_{r,s}(n) = A(iz)\, A(-iz)\, n!\, O(n^2)(|v_r^+ v_{n-r}^-| + |v_{n-s}^+ v_s^-|)$$

as $n \to \infty$. Using Stirling's approximation

$$\Gamma(\alpha + n) = n^{n+\alpha-1/2} e^{-n} \sqrt{2\pi}[1 + O(n^{-1})]$$

as $n \to \infty$ (uniformly for α in a compact set) to estimate v_{n-r}^- and v_{n-s}^- for large n, we get

$$E_{r,s}(n) = A(iz)\, A(-iz)\, C_n [O(n^{-2iz-2r+1}) + O(n^{2iz-2s+1})] \tag{2.6}$$

as $n \to \infty$, provided r and s are sufficiently large (depending on F).

Finally, to obtain the error estimate given in the theorem for any $r, s \geqslant 0$, consider that for some $r' > r$, $s' > s$, (2.6) guarantees

$$E_{r',s'}(n) = A(iz)\, A(-iz)\, C_n [O(n^{-2iz-2r}) + O(n^{2iz-2s})]$$

and that the difference $E_{r,s}(n) - E_{r',s'}(n)$ is a finite sum of terms which are either $\pi_r(iz)\, A(-iz)\, C_n O(n^{-2iz-2r})$ or $\pi_s(-iz)\, A(iz)\, C_n O(n^{2iz-2s})$. This completes the proof.

Asymptotics in the case $z = im/2$ can be obtained as a limiting case of (2.5):

THEOREM. *For integers $m, s \geqslant 0$,*

$$\begin{aligned}P_n(-m^2/4) = &\sum_{k=0}^{m-1} (-1)^k \binom{n}{k} (n+m-2k)\, \pi_k(-m/2)\, \pi_{n-k}(m/2)/(m-k)_{n+1} \\ &+ \frac{(-1)^m}{n!} \sum_{k=0}^{s-1} \binom{n}{k}\binom{n}{m+k} \pi_k(m/2)\, \pi_{n-k}(-m/2) \\ &\times \left\{(n-m-2k) \sum_{j=k}^{n-m-k-1} \left(\frac{1}{a+m/2+j} + \cdots \right.\right. \\ &\left.\left. + \frac{1}{d+m/2+j} - \frac{1}{j+1} - \frac{1}{m+j+1}\right) + 2\right\} \\ &+ \pi_{m+s}(-m/2)\, C_n \ln n O(n^{-m-2s}) \qquad (2.7)\end{aligned}$$

as $n \to \infty$, uniformly for a, b, c, d in compact sets.

Proof. In the previous theorem, put $r = s+m$ and note that the residue of the term $k = m+j$ $(0 \leqslant j \leqslant s-1)$ of the first sum at $z = im/2$ cancels the residue of the term $k = j$ of the second sum. Combining the corresponding terms and calculating the limit as $z \to im/2$ gives the approximation in (2.7). (The same technique also gives a closed form for the exact value of $P_n(-m^2/4)$.) To estimate the error, apply (2.6) in the previous proof, with z on a circle of radius ε centered at $im/2$. This gives, for s sufficiently large,

$$E_{m+s,s}(n) = C_n [\Gamma(a+iz)\, \Gamma(a-iz) \cdots \Gamma(d+iz)\, \Gamma(d-iz)]^{-1}\, O(n^{-m-2s+2\varepsilon}).$$

By the maximum modulus theorem, the same estimate holds at $z = im/2$. The error bound is improved to the one given in the theorem, for all $s \geqslant 0$, by the same technique used at the end of the previous proof.

Note that truncating the first sum of the approximation gives the simpler formula, for $0 \leqslant r \leqslant m-1$,

$$\begin{aligned}P_n(-m^2/4) = &\sum_{k=0}^{r-1} (-1)^k \binom{n}{k} (n+m-2k)\, \pi_k(-m/2)\, \pi_{n-k}(m/2)/(m-k)_{n+1} \\ &+ \pi_r(-m/2)\, C_n O(n^{m-2r}). \qquad (2.8)\end{aligned}$$

Asymptotic expansions in powers of n may be derived by applying Barnes' expansion

$$\Gamma(\alpha+n) \sim n^{\alpha+n-1/2} e^{-n} \sqrt{2\pi}\,[1 + \beta_1 n^{-1} + \beta_2 n^{-2} + \cdots]$$

as $n \to \infty$ (uniform for α in compact sets) to the terms of the expansions (2.5) and (2.7). From (2.5) comes

$$P_n(z^2) = A(-iz)\, C_n \left\{ \sum_{k=0}^{r-1} a_k(z)\, n^{-2iz-k} + O(n^{-2iz-r}) \right\}$$

$$+ A(iz)\, C_n \left\{ \sum_{k=0}^{s-1} a_k(-z)\, n^{2iz-k} + O(n^{2iz-s}) \right\} \tag{2.9}$$

as $n \to \infty$, where $a_0(z) = 1$, $a_k(z)$ is a polynomial in a, b, c, d and an analytic function of z for $z \neq im/2$, and the error bound is uniform on compact sets. This expansion is analogous to known expansions for the classical polynomials [8, various theorems from Theorem 8.21.3 to Theorem 8.22.7]. However, the simplicity and accessibility of the general term of (2.5), as well as the order of the approximation for a given number of terms, strongly recommend it over (2.9).

In a preliminary version of this paper, we derived (2.9) as well as the corresponding expansion for $P_n(-m^2/4)$ by Darboux's method, using the generating function (2.4). (It was (2.9) which suggested the asymptotic character of the ${}_7F_6$ representation).

Darboux's method [7, 8] derives asymptotics for a sequence $\{c_n\}$ from information about the behavior of the generating function $g(w) = \sum_{n=0}^{\infty} c_n w^n$ at the singularities on the circle of convergence. We omit the derivation of (2.9) but note that to prove the uniformity of the error bounds in the expansion it was necessary to use a uniform version of Darboux's theorem:

THEOREM. *If $g(w) = \sum_{n=0}^{\infty} c_n w^n$ is analytic in $\{|w| < 1\}$ and, for some $m \geq 0$, the mth derivative $g^{(m)}(w)$ is continuous on $\{|w| \leq 1\}$, then as $n \to \infty$, $c_n = O(n^{-m})$. Furthermore, if $g(w)$ depends on (complex) parameters $a_1, ..., a_p$, and $g^{(m)}(w)$ is bounded on $\{|w| = 1\}$ uniformly with respect to the a_i, then the estimate $O(n^{-m})$ is also uniform.*

Proof. For $n \geq m$, $0 < r < 1$, Cauchy's theorem says

$$\frac{1}{2\pi i} \int_{|w| = r} \frac{g^{(m)}(w)\, dw}{w^{n-m+1}} = c_n n(n-1) \cdots (n-m+1).$$

In the limit as, $r \to 1-$, we get an integral over the unit circle, and the estimate on c_n is immediate.

The behavior of the generating function (2.4) near the singularity $w = 1$ was obtained by one of Kummer's relations between solutions of the hypergeometric differential equation [3, p. 107, formula (33)], or a limiting version of that formula in case $z^2 = -m^2/4$.

3. Polynomial Series

The formulas of the preceding section are basic tools for attacking problems concerning series $\sum a_n P_n(z^2)$ and their use in approximating functions. Here we restrict ourselves to some results on regions of convergence and analyticity of the sums, obtained by comparison with Dirichlet series $\sum b_n n^{-2iz}$.

First, we introduce some notation and recall some facts from the theory of Dirichlet series. (See [9], for example.) Given a series $\sum_{n=0}^{\infty} a_n P_n(z^2)$, let $a'_n = (2\pi)^{3/2} e^{-3n} n^{3n+a+b+c+d-3/2} a_n$. The series $\sum_{n=1}^{\infty} a'_n n^{-2iz}$ has an ordinate of convergence y_c and an ordinate of absolute convergence y_a, $-\infty \leqslant y_a \leqslant y_c \leqslant +\infty$, such that $\sum a'_n n^{-2iz}$ converges absolutely for $\operatorname{Im} z < y_a$, converges conditionally for $y_a < \operatorname{Im} z < y_c$, and diverges with unbounded partial sums for $\operatorname{Im} z > y_c$. The convergence is uniform for $\operatorname{Im} z \leqslant y_c - \varepsilon$, $\varepsilon > 0$, and the absolute convergence is uniform for $\operatorname{Im} z \leqslant y_a - \varepsilon$. An analog of Hadamard's formula for radius of convergence is the set of inequalities $(-L-1)/2 \leqslant y_a \leqslant y_c \leqslant -L/2$, $L = \limsup_{n \to \infty} \log |a'_n| / \log n$.

For real nonzero y, let $\Omega(y)$ be the domain in the z^2-plane inside the parabola with focus 0 and vertex $-y^2$, and understand $\Omega(\pm\infty)$ to mean the entire plane. If any of the parameters a, b, c, d has a negative real part, say $\operatorname{Re} a < 0$, then there are (finitely many) points $z^2 = -(a+k)^2$ with $k \geqslant 0$ and $\operatorname{Re}(a+k) \leqslant 0$. We denote the set of all such points (considering all four parameters) by Δ, for "discrete spectrum," since these points are mass points in an orthogonality relation for $\{P_n\}$. If the sum of two parameters is an integer less than or equal to zero, say $a+b = -N$, then the points $-(a+k)^2 = -(b+N-k)^2$, $0 \leqslant k \leqslant N$, are mass points in a purely discrete orthogonality relation. We denote the set of all such ponts (considering all six pairs of parameters) by Δ_0.

Theorem. *With the notations just introduced*:

(i) *If* $\operatorname{Im} z > 0$ *and* $z^2 \notin \Delta$, *then* $\sum_{n=0}^{\infty} a_n P_n(z^2)$ *converges absolutely, converges, or has bounded partial sums if and only if* $\sum_{n=0}^{\infty} a'_n n^{-2iz}$ *does.*

(ii) *If* $t > 0$, *then* $\sum_{n=0}^{\infty} a_n P_n(t^2)$ *converges absolutely, converges, or has bounded partial sums if both series* $\sum_{n=0}^{\infty} a'_n n^{\pm 2it}$ *do.*

(iii) *If* $0 \notin \Delta$, *then* $\sum_{n=0}^{\infty} a_n P_n(0)$ *converges absolutely, converges, or has bounded partial sums if and only if* $\sum_{n=0}^{\infty} a'_n \ln n$ *does.*

(iv) *If* $z^2 \in \Delta$, $z^2 \notin \Delta_0$, *then* $\sum_{n=0}^{\infty} a_n P_n(z^2)$ *converges absolutely, converges, or has bounded partial sums if and only if* $\sum_{n=1}^{\infty} a'_n n^{2iz}$ *does.*

(v) *If* $z^2 \in \Delta_0$, *then* $P_n(z^2)$ *is zero for all* n *sufficiently large.*

This theorem tells us for example that if y_c, the ordinate of convergence of the Dirichlet series, is positive, then the series $\sum_{n=0}^{\infty} a_n P_n(z^2)$ converges at all points z^2 in $\Omega(y_c)$ and all points of Δ. If $y_c \leqslant 0$, then it converges at all points of Δ_0 and points of Δ exterior to $\Omega(y_c)$.

Proof. In case (i), one of the asymptotic expansions (2.5) or (2.7) applies. By using enough terms of the expansion we see that $a_n P_n(z^2) = b_n a'_n n^{-2iz}$ with $\{b_n\}_{n=1}^{\infty}$ a sequence of bounded variation. It is also true then, that $|a_n P_n(z^2)| = |b_n|\,|a'_n n^{-2iz}|$ and $\{|b_n|\}_{n=1}^{\infty}$ has bounded variation. Summing by parts establishes that $\sum a_n P_n(z^2)$ converges absolutely, converges, or has bounded partial sums if $\sum a'_n n^{-2iz}$ does.

Now, $\lim_{n\to\infty} b_n = A(iz) \neq 0$, so that $\{1/b_n\}$ also has bounded variation. Therefore, the roles of $a_n P_n(z^2)$ and $a'_n n^{-2iz}$ may be interchanged in the summation by parts argument to prove the "only if" part of the assertion.

The proofs of cases (ii), (iii), and (iv) are similar. Case (v) is more elementary. We may assume that $a + c = -N$ and $a + iz = -j$, $0 \leqslant j \leqslant N$. Then $c - iz = -N + j$. In either representation (2.2) or (2.3) for $P_n(z^2)$, all terms of the sum vanish if $n \geqslant N + 1$.

THEOREM. *If $y_c > 0$, then the convergence of $\sum_{n=0}^{\infty} a_n P_n(z^2)$ is uniform in compact sets in $\Omega(y_c)$. If $y_a > 0$, then $\sum_{n=0}^{\infty} |a_n P_n(z^2)|$ converges uniformly on compact sets in $\Omega(y_a)$.*

Proof. Given a compact set K contained in $\Omega(y_c)$, consider a simple closed curve C in $\Omega(y_c)$ surrounding K and passing through none of the points $-m^2/4$. Since (2.5) is uniform for z^2 on C, and the partial sums of $\sum a'_n n^{-2iz}$ ($\operatorname{Im} z \geqslant 0$) are uniformly bounded for z^2 on C, the partial summation argument used above shows that $\sum a_n P_n(z^2)$ converges uniformly on C. But then by the Cauchy criterion for uniform convergence and the maximum modulus principle, the convergence is uniform on K. The assertion concerning uniform absolute convergence is proved similarly.

4. THE ${}_4\phi_3$ POLYNOMIALS

We list results for the polynomials $p_n((z+z^{-1})/2)$ in (1.6) analogous to those for $P_n(z^2)$. We assume that $0 < |q| < 1$. The proofs are very similar to those in the preceding sections.

If q is real, $0 < |q| < 1$, and a, b, c, d are real or, if complex, occur in conjugate pairs, then $P_n(x)$ is real for real x. If also $|a|, |b|, |c|, |d| < 1$, then there is a real orthogonality relation

$$\int_{-1}^{1} p_n(x)\, p_m(x)\, w(x)\, dx = \delta_{m,n} h_n. \tag{4.1}$$

There are also purely discrete and mixed orthogality relations with mass points at $(z+z^{-1})/2 = (aq^k + a^{-1}q^{-k})/2$. (See [1]).

The representation analogous to (2.2), obtained from (1.6) and Watson's analogs [2] of Whipple's transformations, is

$$
\begin{aligned}
&p_n((z+z^{-1})/2)\\
&\quad=\frac{\pi_n(z^{-1})}{(z^{-2};q)_n z^n}\\
&\quad\quad\times {}_8\phi_7\left[\begin{matrix} z^2q^{-n}, zq^{-n/2+1}, -zq^{-n/2+1}, az, bz, cz, dz, q^{-n}; q, q \\ zq^{-n/2}, -zq^{-n/2}, za^{-1}q^{-n+1}, ..., zd^{-1}q^{-n+1}, qz^2 \end{matrix}\right]\\
&\quad=(q;q)_n \sum_{k=0}^{n} u_k(z)\, u_{n-k}(z^{-1}) \frac{zq^k - z^{-1}q^{n-k}}{z-z^{-1}}\, q^{k(n-k)},
\end{aligned}
\tag{4.2}
$$

where

$$\pi_k(z)=(az;q)_k \cdots (dz;q)_k \qquad \text{and} \qquad u_k(z)=\pi_k(z)/(q;q)_k\,(qz^2;q)_k\, z^k.$$

Again, this expansion has an asymptotic character. If $z \notin \{\pm q^{m/2} : m \in \mathbb{Z}\}$ and $z \neq 0$ then as $n \to \infty$,

$$
\begin{aligned}
p_n((z+z^{-1})/2) = (q;q)_n &\sum_{k=0}^{r-1} u_k(z)\, u_{n-k}(z^{-1}) \frac{zq^k - z^{-1}q^{n-k}}{z-z^{-1}}\, q^{k(n-k)}\\
&+\pi_r(z)\, A(z^{-1})\, O(z^n q^{rn})\\
&+(q;q)_n \sum_{k=0}^{s-1} u_{n-k}(z)\, u_k(z^{-1}) \frac{zq^{n-k} - z^{-1}q^{k}}{z-z^{-1}}\, q^{k(n-k)}\\
&+\pi_s(z^{-1})\, A(z)\, O(z^{-n} q^{sn}),
\end{aligned}
\tag{4.3}
$$

where $\pi_k(z)$ and $u_k(z)$ are as in (4.2),

$$A(z)=(az;q)_\infty\,(bz;q)_\infty\,(cz;q)_\infty\,(dz;q)_\infty/(z^2;q)_\infty,$$

and the O-estimates are uniform for z, a, b, c, d in compact sets. In particular (set $r=s=1$)

$$
\begin{aligned}
p_n((z+z^{-1})/2) &= \frac{(a/z;q)_n\,(b/z;q)_n\,(c/z;q)_n\,(d/z;q_n\, z^n}{(z^{-2};q)_n}(1+O(q^n))\\
&\quad+\frac{(az;q)_n\,(bz;q)_n\,(cz;q)_n\,(dz;q)_n\, z^{-n}}{(z^2;q)_n}(1+O(q^n))\\
&=A(z^{-1})\, z^n(1+O(q^n)) + A(z)\, z^{-n}(1+O(q^n)).
\end{aligned}
\tag{4.4}
$$

In the proof of expansion (4.3), the error term is

$$E_{r,s}(n) = (q;q)_n \sum_{k=r}^{n-s} u_k(z)\, u_{n-k}(z^{-1}) \frac{zq^k - z^{-1}q^{n-k}}{z - z^{-1}} q^{k(n-k)}$$

$$= (1-z^2)\, A(z)\, A(z^{-1})\, O(1) \sum_{k=r}^{n-s} |q|^{k(n-k)}\, |z|^{n-2k}$$

(for r, s sufficiently large, $n \to \infty$). The terms of the latter sum are obviously log convex as a function of k, and the rest of the proof goes as the proof of (2.5).

For $z = \pm q^{\pm m/2}$, $m = 0, 1, 2, \ldots$, we have the q-versions of (2.7) and (2.8):

$$\begin{aligned}
&p_n(\pm(q^{m/2} + q^{-m/2})/2) \\
&= (q;q)_n \sum_{k=0}^{m-1} u_k(\pm q^{-m/2})\, u_{n-k}(\pm q^{m/2}) \frac{1 - q^{n+m-2k}}{1 - q^m} q^{k(n-k+1)} \\
&\quad + \frac{(-1)^m}{(q;q)_n} \sum_{k=0}^{s-1} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ m+k \end{bmatrix}_q q^{m(n-m+1)/2 - k(n-m-k+1)} \\
&\quad \times \pi_k(\pm q^{m/2})\, \pi_{n-k}(\pm q^{-m/2}) \Bigg\{ (1 - q^{n-m-2k}) \\
&\quad + (1 - q^{n-m-2k}) \sum_{j=k}^{n-m-k-1} \left(1 \pm \frac{aq^{j+m/2}}{1 \mp aq^{j-m/2}} \pm \cdots \right. \\
&\quad \left. \pm \frac{dq^{j+m/2}}{1 \mp dq^{j+m/2}} - \frac{q^{j-1}}{1 - q^{j+1}} - \frac{q^{m+j+1}}{1 - q^{m+j-1}} \right) \Bigg\} \\
&\quad + \pi_{m+s}(\pm q^{-m/2})\, n \cdot O(q^{((m/2)+s)n}).
\end{aligned} \tag{4.5}$$

We have used q-binomial coefficient notation:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = (q;q)_n / (q;q)_k\, (q;q)_{n-k}.$$

If the approximation is truncated after the term $k = r - 1$ of the first sum, $0 \leqslant r \leqslant m - 1$, then the error is $\pi_r(\pm q^{-m/2})\, O(q^{((-m/2)+r)n})$.

The analogs of representation (2.3) and the generating function (2.4) appeared in [6], where Darboux's method or termwise limits were used to derive the major terms in the asymptotic expansion. A convergent complete asymptotic expansion was derived in [4].

For the q-versions of the polynomial series results, define Δ to be the set of points $(z + z^{-1})/2$ with $|z| \geqslant 1$, $z = \alpha q^k$, $k \geqslant 0$, $\alpha = a, b, c$, or d. Let Δ_0 be the set of all points $(z+z^{-1})/2$ with $z=\alpha q^k=(\beta q^{N-k})^{-1}$, $0\leqslant k\leqslant N$, where α and β are any two of the four parameters a, b, c, d. (Δ_0 is a subset of Δ and is empty unless some pair α, β has product q^{-N}, $n\geqslant 0$.)

THEOREM. *With the notation just introduced:*

(i) *If* $\operatorname{Im} z>0$ *and* $(z+z^{-1})/2\notin\Delta$, *then* $\sum_{n=0}^{\infty} a_n p_n((z+z^{-1})/2)$ *converges absolutely, converges, or has bounded partial sums if and only if* $\sum_{n=0}^{\infty} a_n z^n$ *does;*

(ii) *If* $0<\theta<\pi$, *then* $\sum_{n=0}^{\infty} a_n p_n(\cos\theta)$ *converges absolutely, converges, or has bounded partial sums if both series* $\sum_{n=0}^{\infty} a_n e^{\pm in\theta}$ *do.*

(iii) *If* $1\notin\Delta$ (*or alternatively if* $-1\notin\Delta$), *then* $\sum_{n=0}^{\infty} a_n p_n(1)$ (*or* $\sum_{n=0}^{\infty} a_n p_n(-1)$) *converges absolutely, converges, or has bounded partial sums if and only if* $\sum_{n=0}^{\infty} n a_n$ *does.*

(iv) *If* $(z+z^{-1})/2\in\Delta$, *and* $(z+z^{-1})/2\notin\Delta_0$, *then* $\sum_{n=0}^{\infty} a_n p_n((z+z^{-1})/2)$ *converges absolutely, converges, or has bounded partial sums if and only if* $\sum_{n=0}^{\infty} a_n z^{-n}$ *does.*

(v) *If* $(z+z^{-1})/2\in\Delta_0$, *then* $p_n((z+z^{-1})/2)$ *is zero for all n sufficiently large.*

According to this theorem, if $\sum a_n z^n$ has radius of convergence $\rho>1$, then $\sum a_n p_n((z+z^{-1})/2)$ converges at all points $(z+z^{-1})/2$ in E_ρ, the ellipse with foci ± 1 and vertices $\pm(\rho+\rho^{-1})/2$, and at all points of Δ. If $\rho\leqslant 1$, then it converges at all points of Δ exterior to E_ρ as well as at all points of Δ_0.

THEOREM. *If* $\sum a_n z^n$ *has radius of convergence* $\rho>1$, *then* $\sum |a_n p_n((z+z^{-1})/2)|$ *converges uniformly on compact sets in* E_ρ.

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